LIMITATIONS TO EXTENDED STATIONARY CORONAE

AN ANALYTICAL MODEL

Hans van der Meer

Sterrekundig Instituut, Zonnenburg 2, 3512 NL Utrecht, The Netherlands

SUMMARY

A simple analytical model for stellar corona is developed. It is based on the assumption of an isothermal stationary corona and a transition region of constant pressure.
For mathematical convenience it is assumed that the dissipation of mechanical energy takes place in a layer of negligible vertical extend.
It is shown that there is an upper limit for the mechanical flux heating the corona. If this limit is exceeded no extended stationary corona is possible. It is suggested that a thin corona is formed.
The results of the calculations are in good agreement with the numerical results of Hearn and Vardavas (1981).
It is shown that there is a lower bound to the height of the dissipative layer above the chromosphere.

Key words: transition region, stellar coronae, solar coronae.

1. INTRODUCTION

Several authors (Hearn and Vardavas (1981), Hammer (1981), Souffrin (1982) and Hearn et al. (1983)) have found theoretical indications that there exists an upper limit for the mechanical flux heating a corona. Hearn and Vardavas (1981) have suggested that a thin hydrostatic corona is formed if the mechanical flux exceeds this upper limit.
In this paper an analytical for a stellar corona is presented. In this model the energy balance in the transition region is examined. This energy balance is determined by the heating of the corona, the energy losses due to radiation and the divergence of the conducted heat. This balance can be put in the following form

$$\nabla \cdot \underline{J} = q_h - q_r \tag{1}$$

where q_h is the mechanical heating of the corona, q_r is the radiation loss and $\underline{J}$ is the conducted heat. (See Martens (1981))
For a given heating function there is only one solution of (1) consistent with the boundary condition at infinity.
(1) will be solved using heating functions in a mathematical convenient form. The heating function is specified by two parameters. The aim of this work is to find out which combinations of parameters yield extended stationary coronae.
To compare the results with the results of Hearn and Vardavas (1981) the calculations will be done for an OB- supergiant.
It will be confirmed that there is a maximum mechanical flux consistent with an extended stationary corona, and according to the numerical result of Hearn and Vardavas (1981) a solution for a thin hydrostatic corona is found.
Finally the theory will be applied to the solar corona.

2. THE CORONAL MODEL

2.1 The energy equation

This section describes the energy equation which is valid in the transition region. The following assumptions are being made: (Martens 1981)

- The thickness of the transition region is small compared to the stellar radius, so a plane parallel geometry may be used.
- The thickness of the transition region is smaller than the pressure scale height, so constant gas pressure is assumed.
- The stellar wind velocity and energy are negligible.
- The atmosphere consists of fully ionized hydrogen.
- The transition is homogeneous in the horizontal direction, so the model can be one dimensional.

With the assumptions above the gas law can be written as

$$P_o = 2\, n_e\, kT \tag{2}$$

where P_o is the constant gas pressure, n_e the electron density and T is the temperature. k is Boltzmann's constant.

The radiation losses of a hot ionized, optically thin gas of stellar composition can be described by

$$q_r = n_e^2\, j(T) \tag{3}$$

(Cox and Tucker 1969, McWhirter et al. 1975)

For $1.5\ 10^4\ K < T < 10^6\ K$ $j(T)$ can be approximated within a factor two by

$$j_o(T) = 1.8\ 10^{-22} \quad (\mathrm{erg\ cm^3\ s^{-1}}) \tag{4}$$

The divergence of the conducted heat is given by

$$\nabla . J = - \frac{d}{dh} \left(\kappa_o\, T^{5/2} \frac{dT}{dh} \right) \tag{5}$$

where h is the height in the transition region.

An estimate for κ_o is given by Athay (1971, p. 36)

$$\kappa_o = 1.1\ 10^{-6} \quad (\mathrm{erg\ cm^{-1}\ s^{-1}\ K^{-7/2}}) \tag{6}$$

In the calculations the heating function q_h will be a nonnegative function, which is zero beyond some height h_o. The function satisfies

$$\int_0^{h_o} q_h(h)\, dh = F_o \tag{7}$$

where F_o is the total mechanical flux entering the transition region. The following heating function will be used in this paper

$$q_h(h) = F_o\, \delta(\, h - L_o\,) \tag{8}$$

where $\delta(x)$ is the Dirac delta function.

This heating function represents very intens heating at one place L_o. This will be called δ-heating. Physically this means that the mechanical flux is dissipated in a very thin layer around L_o. Combining (2), (3), (5) and (8) yields

$$\kappa_o \frac{d}{dh} \left(T^{5/2} \frac{dT}{dh} \right) = \frac{j_o\, P_o^2}{4\, k^2\, T^2} - F_o\, \delta(\, h - L_o\,) \tag{9}$$

This is a second order, non linear differential equation for the temperature. The boundary conditions are $T(0) = T_{chrom}$ and $\frac{dT}{dh}(0) = 0$. The last condition means that there is no thermal conduction down into the chromosphere.

In some cases another heating function will be used; the socalled constant heating (Martens 1981)

$$q_h(h) = \frac{F_o}{L_o} H(\, L_o - h\,) \tag{10}$$

where $H(x)$ is the Heaviside stepfunction.

2.2 Solution of the energy equation in the case of δ-heating

Equation (9) can be made dimensionless by choosing new variables and y.

$$\eta = (\, T / T_a\,)^{7/2} \tag{11}$$

$$y = h / \varepsilon L_o \tag{12}$$

T_a is a reference temperature defined by

$$\frac{j_o\, P_o^2}{4\, k^2\, T_a^2} = \frac{F_o}{L_o} \tag{13}$$

and ε is given by

$$7\, \varepsilon^2 = \kappa_o\, T_a^{7/2} / L_o F_o \tag{14}$$

The result is

$$2\,\frac{d^2\eta}{dy^2} = \eta^{-4/7} - \delta(\,\varepsilon y - 1\,) \tag{15}$$

The boundary conditions become $\eta(0)\;\; 1$ and $\frac{d\eta}{dy}(0) = 0$.
By integration of (15) one easily sees that

$$\lim_{y\uparrow\varepsilon} \frac{d\eta}{dy} - \lim_{y\downarrow\varepsilon} \frac{d\eta}{dy} = 1\,/\,2\varepsilon \tag{16}$$

This jump in the derivative of η is due to the mathematical properties of the Dirac delta function, which are not of physical interest, since the heating will take place in some finite interval.
For $y < 1/\varepsilon$ and $\eta(0) = 0$ the solution of (15) is

$$\eta(y) = \left(\frac{11}{2\sqrt{21}}\;y\right)^{14/11} \tag{17}$$

Because $\eta(1/\varepsilon)$ has to represent a temperature maximum ε has to be less than 0.201. This follows from (16) and (17). A physical expanation for this is that if ε exceeds 0.201 all the dissipated mechanical energy is conducted into the transition region, where it is radiated away. This means there is no enrgy supply to the corona.
For $y > 1/\varepsilon$ the first integral of (15) is

$$\left(\frac{d\eta}{dy}\right)^2 = \frac{7}{3}\eta^{3/7} + c \tag{18}$$

where c is an integration constant.
The condition imposed by (16) and the continuity of η give

$$c = \frac{1}{4\varepsilon^2} - \frac{7}{3}\left(\frac{11}{2\sqrt{21}}\right)^{3/11} \varepsilon^{-14/11} \tag{19}$$

One now can distinguish three cases: $c > 0$, $c < 0$ and $c = 0$.

1. $c > 0$, $\varepsilon < 0.078$
   this gives

$$\frac{d\eta}{dy} = -\left(\frac{7}{3}\eta^{3/7} + c\right)^{\frac{1}{2}} < -\;c^{\frac{1}{2}} \tag{20}$$

   This is not an acceptable solution, because η falls down very rapidly and becomes less than zero.

2. $c < 0$, $0.078 < \varepsilon < 0.201$ (See fig. 1)
   In this case the region beyond the temperature maximum can be approximated fairly well as isothermal. One finds a solar type corona with a modest massloss. Martens (1981) has calculated coronal models using expressions for the energy losses that were derived by Hearn (1975).
   The energy flux entering the corona is

$$F = F_o - \int_0^{L_o} q_r(h)\, dh = (1 - 3.21\,\varepsilon^{8/11})\, F_o \qquad (21)$$

   This will be further discussed in section 3.
3. $c = 0$, $\varepsilon = 0.078$ (See fig. 1)
   The exact solution of (15) is for $1/\varepsilon < y < 2/\varepsilon$

$$\eta(y) = (\eta_{max}^{11/14} - \frac{11}{2\sqrt{21}} (y - 1/\varepsilon))^{14/11} \qquad (22)$$

   In this case one can calculate that

$$\int_0^{2L_o} q_r(h)\, dh = F_o \qquad (23)$$

   so all the dissipated energy is radiated away. The solution representsa thin corona. Beyond $h = 2\,L_o$ the atmosphere will be in radiative equilibrium, since no additional heating is present.

Finally, in the transition region the follwing relations can be derived for the gas pressure and the maximum temperature

$$P_o = 2\,k\,j_o^{-\frac{1}{2}}\,(7/\kappa_o)^{2/7}\,L_o^{-3/14}\,F_o^{11/14}\,\varepsilon^{4/7} \qquad (24)$$

$$T_{max} = (11/2\sqrt{21})^{4/11}\,(7/\kappa_o)^{2/7}\,(L_o F_o)^{2/7}\,\varepsilon^{16/77} \qquad (25)$$

These expressions will be used in section 3.

2.3 The possibility of a thin corona

The case $c = 0$ can be interpreted as a thin corona. Numerical calculations of Hearn and Vardavas (1981) and Vardavas and Hearn (1981) have been done for stationary stellar coronae. The calculations concerned an OB supergiant with an effective temperature of 31000 K, a mass of 44.7 solar masses and a radius of 27.8 solar radii. The effect of

electronscattering radiation pressure was included by reducing the effective stellar mass to 25.108 solar masses.
These coronae were heated by sawtooth waves with a period of 17000 seconds. The mechanical flux was specified deep in the photosphere. Fluxes of 10^3, 10^4 and 10^5 erg cm^{-2} s^{-1} yielded extended stationary coronae. However, with a flux of 10^6 erg cm^{-2} s^{-1} the densities in the outer regions of the corona became so high that thermal conduction could no longer maintain the high temperatures. The corona collapsed and a thin hydrostatic corona was formed. This corona had a thickness of only 0.03 stellar radius.
The results of Vardavas and Hearn (1981) and of the calculations done in section 2.2 are shown in table 1.

	δ- heating	constant heating	Vardavas and Hearn (1981)
P_o (dyne cm^{-2})	$7.1\ 10^{-3}$	$6.3\ 10^{-3}$	$6.71\ 10^{-3}$
T_{max} (K)	$9.8\ 10^5$	$7.0\ 10^5$	$6.98\ 10^5$

TABLE 1. Comparison with the calculations of Vardavas and Hearn (1981). The pressure and maximum temperature of a thin corona of an OB supergiant. The mechanical flux entering the transition region is $2.5\ 10^4$ erg $cm^{-2}s^{-1}$. The thickness of the corona is $0.03\ R = 5.8\ 10^{10}$ cm. The solution achieved for constant heating is the one that satisfies $T(0) = T(L_o)$.

From table 1 it is clear that the analytical results are in good agreement with the numerical result of Vardavas and Hearn.
The effect of the type of heating function on the coronal pressure is not very important. Only the temperature varies rather strongly. The reason one finds a maximum temperature that is higher in the case of δ-heating is that the dissipation of mechanical enrgy takes place in a very thin layer.
The question that arises is: when is an isothermal model for a corona applicable and when is a thin corona possible? In other words: what are the limitations to extended stationary coronae?
This question will be answered in the next section.

3. LIMITATIONS TO EXTENDED STATIONARY CORONAE

In the case $c < 0$ the region beyond the temperature maximum can be approximated to be isothermal.

For a given mechanical flux F_o and a lenght L_o a whole set of combinations P_o and T_{max} are found. Only one of these is consistent with the boundary condition at infinity (Martens 1981).

Hearn (1975) derived expressions for the energy losses by radiation and stellar wind for a corona that is isothermal up to the Parker critical point and beyond that expands adiabatically.

These expressions are

$$F_W = 6.18\ 10^7\ M^{\frac{1}{2}}\ R^{-\frac{1}{2}}\ P_o\ x^{\frac{1}{2}} \exp(\ 1.33 - 1.16\ x - 0.0364\ x^2\ ) \qquad (26a)$$

$$F_R = 2.84\ 10^5\ M^{-3}\ R^4\ P_o^2\ \ (\ 0.208\ x^2 + 1.03\ x - 1.27\ ) \qquad (26b)$$

$$x = 5.78\ 10^6\ M\ R^{-1}\ T_{max}^{-1}$$

The energy losses due to radiation F_R and due to stellar wind F_W are given in erg $cm^{-2}\ s^{-1}$. M and R are the stellar mass and radius in solar units.

The flux that is available for these losses is given by (21).

The equation to solve is

$$F_{in} = F_{out} \qquad (27)$$

or

$$F_o(\ 1 - 3.21\,\varepsilon^{8/11}\ ) = F_R + F_W \qquad (28)$$

Together with the expressions (24) and (25), which are for given F_o and L_o funtions of ε alone, (28) yields a value for ε.

It may be clear that if one finds a value $\varepsilon \simeq 0.078$ the isothermal approach is no longer valid. In that case the temperature fall back to photospheric levels. It is then that a stationary, extended corona cannot exist.

It is suggested that a thin hydrostatic corona is formed.

4. RESULTS

4.1 Results for an OB supergiant

The theory of sections 2 and 3 was applied to an OB supergiant with an effective mass of 25 solar masses and a radius of 28 solar radii. Coronal models were calculated for increasing mechanical flux F_o, while L_o was kept constant at 2.9 10^{10} cm.

F_o	ε	P_{-3}	T_6	F_W	F_R	n_{e7}
10000	0.12	4.3	0.82	340	2800	1.4
20000	0.10	6.9	0.97	2400	5300	2.5
30000	0.09	8.8	1.06	5800	7300	2.9
40000	0.08	10.3	1.12	10000	9200	3.2
43500	0.078	10.8	1.14	12000	9800	3.6

TABLE 2. Results for an OB supergiant, with an effective mass of 25 $M_\odot$ and a radius of 28 $R_\odot$. L_o = 2.9 10^{10} cm. All quantities in c.g.s. units. (X_n means X x 10^n)

Table 2 shows that if the mechanical flux increases the electron density in the region of the temperature maximum increases too. The electron density in that region is given by

$$n_e = j_o^{-\frac{1}{2}} \left(2\sqrt{21}/11 \right)^{4/11} \varepsilon^{4/11} \left(F_o / L_o \right)^{\frac{1}{2}} \tag{29}$$

This means that the local radiation losses are proportional to F_o / L_o. For a mechanical flux greater than about 4 10^4 erg cm^{-2} s^{-1} a stationary extended corona can no longer exist. Just before the corona collapses the mass loss is 5 10^{-12} M_o y^{-1}.

Hearn et al. (1983) found in their iterative calculations, that before the corona collapsed the mass loss was 2 10^{-12} M_o y^{-1}. They found a maximum mechanical flux of 4.3 10^4 erg cm^{-2} s^{-1}, a pressure in the transition region of 1.3 10^{-2} dyne cm^{-2} and a maximum temperature of 1.8 10^6 K.

One has to realise that before the corona collapses the energy losses due to stellar wind are as important as the losses due to radiation. Locally however, that is at the base of the corona, the radiation losses are much more important than the stellar wind losses.

This means that the model, which neglect stellar wind effect in the transition region, is still applicable.
Figure 2 shows the maximum mechanical flux F_o, consitent with a stationary extended corona, versus the height L_o of the dissipative layer. The maximum mechanical flux is about 5 10^4 erg cm^{-2} s^{-1}.
From figure 2 it is also clear that there seems to exist a minimum L_o consistent with a stationary extended corona. This will be discussed in section 5.

4.2 The solar corona

The theory of sections 2 and 3 was also applied to the solar corona. Using a δ-heating function the maximum mechanical flux is about 2 10^6 erg cm^{-2} s^{-1}.
Hammer (1981) found in his numerical calculations no solutions for a mechanical flux greater than 3 10^5 – 1 10^6 erg cm^{-2} s^{-1}. This maximum depended on the dissipation lenght of the mechanical flux.
To compare the calculations with the results of Martens (1981) and to investigate the influence of the type of heating function on the results, calculations were done with the constant heating function.
The major difference, as can be seen from figure 4, is that the maximum mechanical flux is larger: 5 10^6 erg cm^{-2} s^{-1}. The resemblance is that in both cases a minimum L_o is found. (Note the difference in the definition of L_o for the both heating functions.)
In figure 4 the models of Martens are plotted. One of his models lies outside the region in the L_o - F_o plane where stationary extended coronae are possible.

5. THE INFLUENCE OF L_o

Figures 2, 3 and 4 indicate that there exists a minimum value of L_o. For a value less than this minimum no stationary extended corona is possible. This is a new result. It can be made clear by examining (21), (24), (25) and (26). When $\varepsilon = 0.078$ for the solar corona is found

$$f_W = F_W / F_o = 7.8\ 10^6\ (L_o\ F_o)^{-5/14}\ \exp(\ 1.33 - 1.16\ 10^5\ (L_o\ F_o)^{-2/7} - 3.36\ 10^8\ (L_o\ F_o)^{-4/7}) \tag{30a}$$

$$f_R = F_R / F_o = 9\ 10^7\ L_o^{-1} + 4.5\ 10^3\ L_o^{-5/7}\ F_o^{2/7} - 5.8\ 10^{-2}\ L_o^{-3/7}\ F_o^{4/7} \tag{30b}$$

For large values of L_o wind losses become important, but for smaller values of L_o they are negligible compared to radiation losses.
To find a minimum value one has to examine (30b). (30b) has three terms. The physical of the first term of (30b) may be obvious. It describes the radiation losses at the base of the corona, since it was shown that these losses are proportional to F_o/L_o.
The second and third term describe the radiation losses in the corona further away from the star. One can easily check that the sum of these two term is positive in all cases of physical interest.
The line drawn in figure 3 can be calculated from the equation

$$f_W + f_R = \tfrac{1}{2} \tag{31}$$

For the solar corona this yields a minimum value of L_o

$$L_{o_{min}} = 1.8\ 10^9\ (\ \mathrm{cm}\ ) \tag{32}$$

In general one has

$$L_{o_{min}} = 1.8\ 10^9\ M^{-1}\ R^2\ (\ \mathrm{cm}\ ) \tag{33}$$

where M and R are the stellar mass and radius in solar units.

6. DISCUSSION AND CONCLUSIONS

Souffrin (1982) found an upper limit for the mechanical flux. For a larger mechanical flux no stationary dynamical corona was possible. The heating mechanism he used was dissipation of shockwaves. He derived relations between the mechanical flux, the coronal temperature and the radiation loss of an extended isothermal corona. He then stated that the total radiation loss could not exceed the total mechanical heating. This yielded a maximum mechanical flux consistent with an extended isothermal corona. He did not take into account the energy losses due to the stellar wind, and he did not indicate what was going to happen if the maximum mechanical flux was exceeded.

In this paper it is confirmed that there is an upper limit for the mechanical flux. It is shown that in some cases no extended corona is possible even if the total radiation loss less then 25 % of the total heating.

An estimate is given for the upper limit of the mechanical flux and it is suggested that if this upper limit is exceeded a thin, hydrostatic corona is formed. The stability of this thin corona is not investigated. For this a more refined analysis is needed. In such an analysis one has to account for the interactions between the corona and the photosphere. Hearn et al. (1983) suggest that the corona will undergo a relaxation oscillation.

It is granted that the coronal models in this paper are simplifications of reality. Especially for the solar corona, where magnetic fields are very important.

However, the advantages of such a simple model are obvious. The models are completely analytical. In an easy way one gains insight in some important properties of stellar corona.

REFERENCES

Athay, R.G.: 1971, Physics of the solar corona, Ed. C.J. Macris, Reidel, Dordrecht.

Cox, D.P. and Tucker, W.H.: 1969, Astrophys. J. 157, 1159.

Hammer, R.: 1981, Structure, energy balance and mass loss of stellar coronae, Thesis, Würzburg.

Hearn, A.G.: 1975, Astron. Astrophys. 40, 355.

Hearn, A.G., Kuin, N.P.M., Martens, P.C.H.: 1983, Astron. Astrophys. 125, 69.

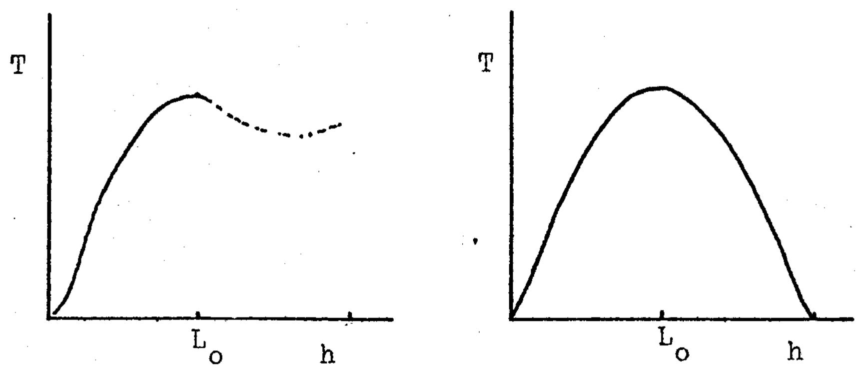
Hearn, A.G. and Vardavas, I.M.: 1981, Astron. Astrophys. 98, 230.

Martens, P.C.H.: 1981, Astron. Astrophys. 102, 156.

Mc-Whirter, R.W.P., Thonemann, P.C., Wilson, R.: 1975, Astron. Astrophys. 40, 63.

Souffrin, P.: 1982, Astron. Astrophys. 109, 205.

Vardavas, I.M. and Hearn, A.G.: 1981, Astron. Astrophys. 98, 241.



(a) $c < 0$ (b) $c = 0$

Figure 1: SOLUTION OF THE ENERGY EQUATION

For $c < 0$ the solution of the energy equation is shown in (a). For $h > L_o$ one has to match the pressure and temperature to the isothermal corona energy loss expressions.

(b) represents the thin corona solution.

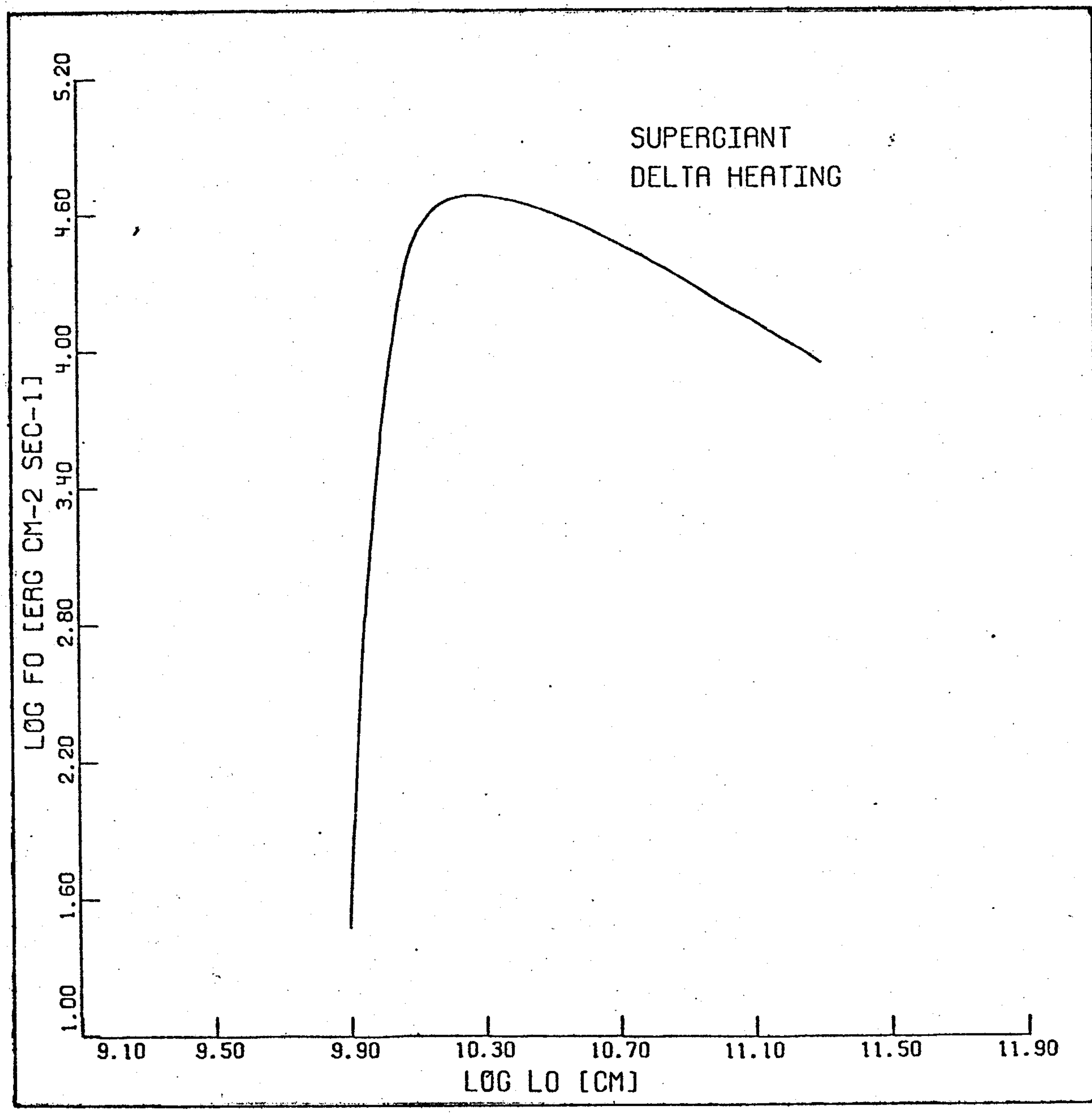


Figure 2: OB SUPERGIANT, models with δ-heating.
Maximum mechanical flux versus typical length scale for a typical OB supergiant, $M_{eff} = 25\ M_{\odot}$, $R = 28\ R_{\odot}$. The maximum mechanical flux consistent with an extended stationary corona is about $5\ 10^4$ erg cm^{-2} s^{-1}.

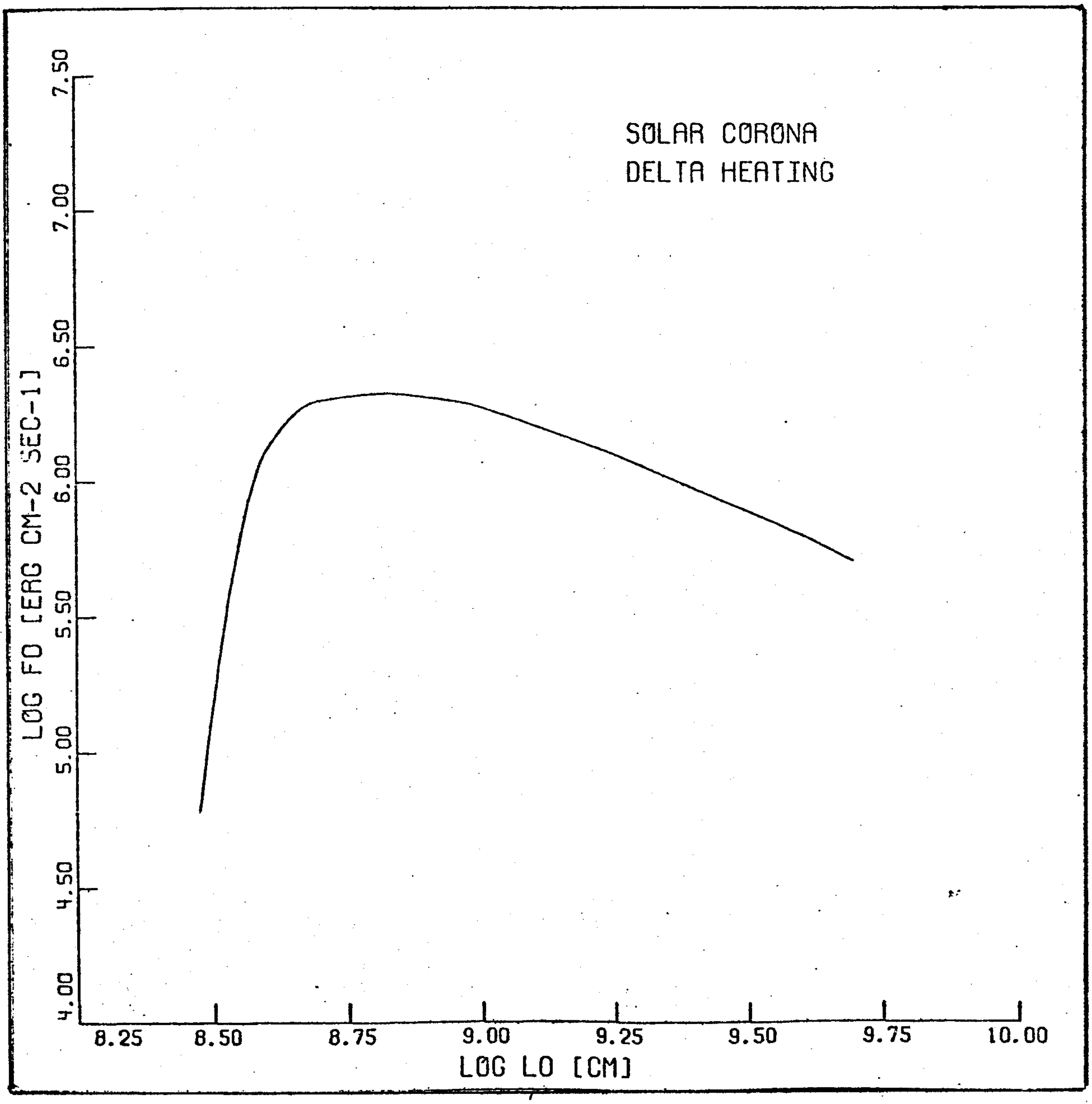


Figure 3: SOLAR CORONA, models with δ-heating.
Maximum mechanical flux versus typical length scale. The maximum mechanical flux consistent with an extended stationary corona is about 2 10^6 erg cm^{-2} s^{-1}.

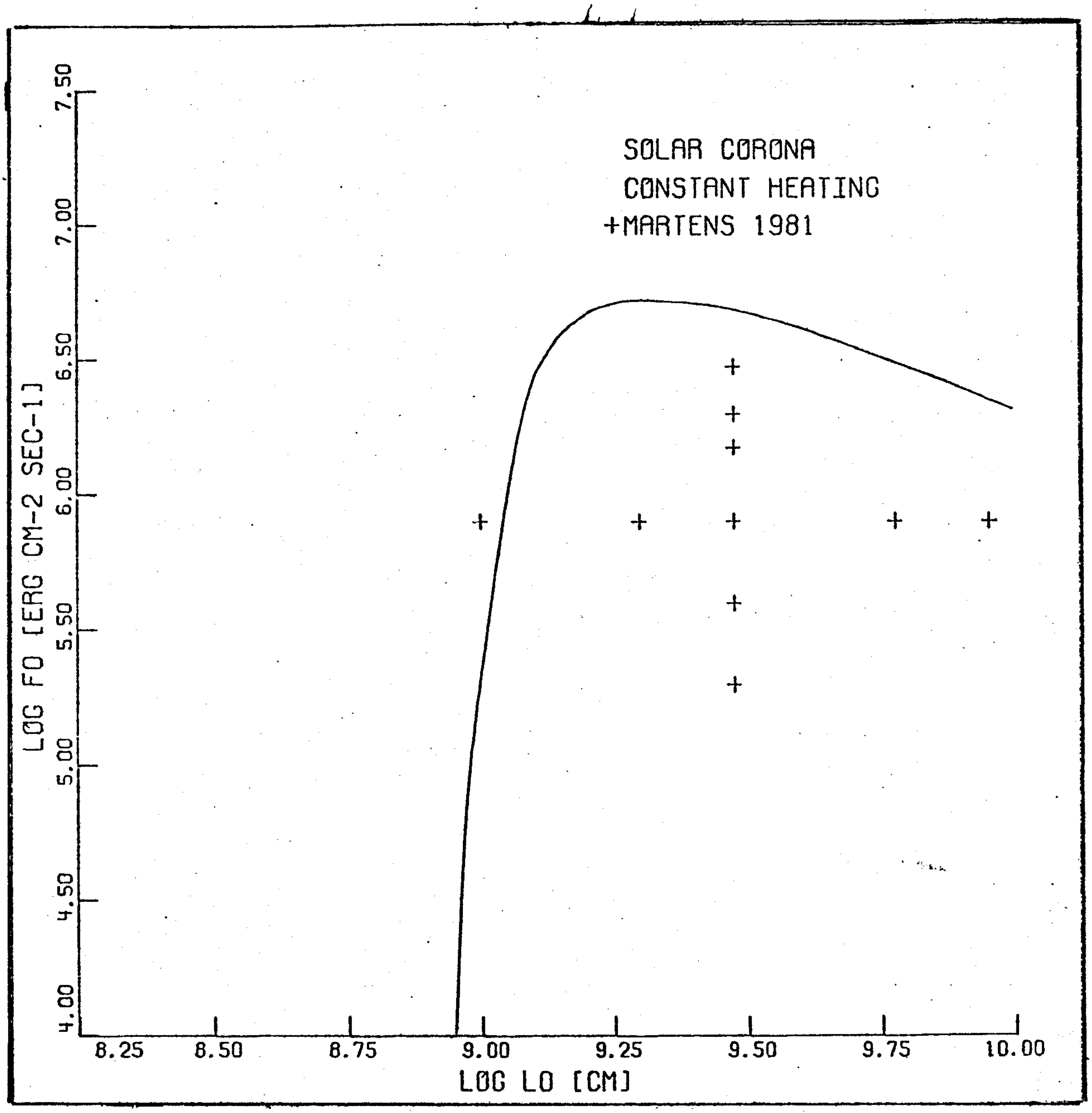


Figure 4: SOLAR CORONA, models with constant heating.
Maximum mechanical flux versus typical length scale.
The crosses represent the models of Martens (1981). One of his models lies outside the region where extended stationary coronae are possible.